Stability Analysis of a Class of Diffusion Flames

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The stability of a particular class of laminar diffusion flames is analytically investigated. The combustion situation consists of two axially symmetric flame sheets in a coaxial flow of a fuel-containing annulus surrounded by flows of oxidizer. At issue is whether or not such flames can be stable. The flames are considered to be geometrical surfaces and a linear stability analysis for axisymmetric disturbances is carried out in the absence of viscosity. Further, the effects of unsteady mass and energy diffusion and unsteady energy conduction are neglected. Under these conditions, it is found that axisymmetric diffusion flames amplify almost all axisymmetric disturbances.

Nomenclature

a	$=c/b$, nondimensional ratio of tube dimensions, a measure of the "thinness" of the fuel zone, sandwiched between the two oxidizer zones (larger values of a imply "thinner" fuel zones)
A	$=\bar{\rho}\bar{D}$
$A_1,A_2,A_3,$ A_4,A_5	= complex constants
b,c	= tube dimensions
B	= nondimensional constant related to initial condition of oxidizer
c_P	= specific heat at constant pressure of the mixture
C°	= complex constant, defined in discussion of asymptotic solutions
D	= binary diffusion coefficient for all pairs of species (assumed equal)
E	= nondimensional constant related to initial condition of fuel
$f_x(r)$	= amplitude functions; eigenfunctions, functions of r also weakly dependent on z
(f_{P_r}, f_{P_i})	= real and imaginary components of f_P
F	= nondimensional α_z ($1/F$ is a measure of the wavelength of the disturbance in the axial direction; F is purely imaginary)
F_i	$=F/i$
G	= nondimensional α_t; eigenvalue of the problem
(G_r,G_i)	= real and imaginary parts of G [the real part of G is a measure of the growth (or decay) of the disturbances; the imaginary part of G can be used to obtain the velocity of propagation (phase velocity) of the disturbances]
h_j°	= standard heat of formation per unit mass for species j at temperature T°
$H_0^{(1)},H_0^{(2)}$	= Hankel functions of order zero
i	$=\sqrt{-1}$
J_0,J_1	= Bessel functions of first kind of orders zero and one
K	= scaling factor
M	= measure of the square of the initial Mach number
M_j	= symbol for chemical species j
n	= dimensionless integer
N	= total number of chemical species present
P	= hydrostatic pressure
P_e	= Peclet number
q	= variable of integration
r,θ,z	= cylindrical coordinates
R	= nondimensional parameter that is a measure of the heating value of the chemical reaction
R°	= universal gas constant
S	$=\bar{\rho}\bar{w}$
t	= time
T	= temperature
T°	= standard reference temperature
T_s	= initial temperature of fuel and oxidizer streams
u,v,w	= components of mass average velocity in the r,θ,z directions
V	= nondimensional $\bar{w}$
V	= mass average velocity of the gas mixture
V_0	= characteristic velocity of the flow
V°	= value of V near $\xi=0$
W	= average molecular weight of the gas mixture
W_j	= molecular weight of species j
x	= general variable used to represent any of ρ,u,v,w,P,β, and τ
y	= general variable used to represent either β or τ
Y_0	= Bessel function of the second kind of order zero
Y_j	= mass fraction of species j
$Y_{ox,0}$	= mass fraction of oxidizer in the oxidizer stream at $z=0$
$Y_{f,0}$	= mass fraction of fuel in the fuel stream at $z=0$
Z	= complex variable
α_f	$\equiv Y_f/W_f(\nu_f''-\nu_f')$
α_{ox}	$\equiv Y_{ox}/W_{ox}(\nu_{ox}''-\nu_{ox}')$
α_T	$\equiv \int_{T^\circ}^{T} c_P \mathrm{d}T \Big/ \sum_{j=1}^{N} h_j^\circ W_j(\nu_j'-\nu_j'')$
α_z	= imaginary number
α_t	= complex number
α_{zi}	$=\alpha_z/i$
$(\alpha_{tr},\alpha_{ti})$	= real and imaginary parts of α_t
β	$=\alpha_{ox}-\alpha_f$
γ	= ratio of specific heats of the gas mixture
η	$=(z/b)$, nondimensional z coordinate

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ν_j', ν_j'' = stoichiometric coefficients for species j appearing as a reactant or product, respectively, in the one-step reaction

$$\sum_{j=1}^{N} \nu_j' M_j \rightarrow \sum_{j=1}^{N} \nu_j'' M_j$$

ν_f, ν_{ox}, ν_j = stoichiometric coefficients for fuel, oxidizer, and product species j, respectively, in the one-step reaction

$$\nu_f M_f + \nu_{ox} M_{ox} \rightarrow \sum_{j=1}^{N-2} \nu_j M_j$$

ξ = (r/b), nondimensional r coordinate
ρ = overall density
σ = 0 or 1, depending on whether y represents β or τ, respectively
τ = $\alpha_T - \alpha_f$

Subscripts

f = fuel
Im( ) = imaginary part of ( )
ox = oxidizer
Real( ) or Re( ) = real part of ( )

Superscripts

$\bar{(\ )}$ = steady-state value
$(\ )'$ = fluctuation from steady state

Introduction

CLASSICAL theories of diffusion flames are described by steady-state analytical models.[1] However, during the edge burning of laminates consisting of alternate layers of solid oxidizer [ammonium perchlorate (AP)] and solid fuel (e.g., polybutadiene acrylonitrile rubber), it was observed that a stable flame sheet rarely resulted[2]; instead, arrays of flickering flamelets (typically of less than 0.5 ms duration) occurred. In order to study this phenomenon further, an analytical investigation of the stability of diffusion flames resulting from coaxial axisymmetric oxidizer and fuel flows (Fig. 1) was undertaken.

Figure 1 can be considered an approximate representation of the gas-phase geometry just above an AP particle at the surface of a deflagrating solid propellant. Instability in such a flow-flame situation can result from: 1) velocity and density differences between the fuel and oxidizer streams, 2) inviscid hydrodynamic instability, 3) instability in the flow due to the presence of the flame, 4) viscosity-generated instability, 5) exothermic condensed-phase reactions and phase transition phenomena, 6) complex chemical kinetic phenomena, and 7) various other causes. Analytical and experimental studies on the stability of pure jets[3-5] have established that, depending on the initial velocity profile, these jets become unstable for Reynolds numbers in the range of 35-300. Near the critical Reynolds number, the instability is of the inviscid hydrodynamic type and viscosity acts to damp the disturbances. Despite the fact that the flow geometry chosen here (Fig. 1) is different from that of a pure jet, it is entirely possible that unsteady flame behavior is a consequence of this kind of hydrodynamic instability. However, it was decided to analytically explore whether a flame internal to the flow can amplify disturbances. As the flame is a much more potent source of energy to the disturbances than a mere mean flow, such an analysis is certainly justified.

The analysis predicts that diffusion flames resulting from coaxial, axisymmetric oxidizer and fuel flows can amplify axisymmetric disturbances for any Peclet number that is large compared to unity (say 10). As the Reynolds and Peclet numbers are of comparable magnitude, it follows that instability due to the presence of the flame is at least as important as hydrodynamic instability. Within the scope of the present analysis, one cannot answer the question: Which of the two instabilities is more critical? However, as mentioned, for the flow regime under consideration, the flame is a far more potent source of energy to the disturbances than a pure mean flow. Consequently, it may be expected that instability due to the presence of the flame is the dominant factor.

The analysis is only for gas-phase processes and does not consider the solid-phase aspect of the problem. The reason for this is that the gas-phase flame is the most obvious source of energy for gas-phase disturbances. Furthermore, if the frequency of the disturbances is large enough, the condensed phase is not likely to play a part in the growth or decay of the disturbances. The effect of viscosity has also been neglected. This approximation can be justified on the grounds that for the Reynolds numbers under consideration (10-100), viscosity acts to damp the disturbances. Consequently, its inclusion will not change any qualitative conclusions that can be obtained regarding instability due to the presence of a flame. Its inclusion will, of course, change the numerical values of amplification and frequency for the disturbances.

Governing Equations

In the absence of viscosity, the continuity and momentum equations for a compressible fluid are

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial r} + \frac{\rho u}{r} + \frac{1}{r}\frac{\partial \rho v}{\partial \theta} + \frac{\partial \rho w}{\partial z} = 0 \tag{1}$$

$$\rho\frac{\partial u}{\partial t} + \rho\frac{u\partial u}{\partial r} + \frac{\rho v}{r}\frac{\partial u}{\partial \theta} + \rho w\frac{\partial u}{\partial z} - \frac{\rho v^2}{r} = -\frac{\partial P}{\partial r} \tag{2}$$

$$\rho\frac{\partial v}{\partial t} + \rho u\frac{\partial v}{\partial r} + \frac{\rho v}{r}\frac{\partial v}{\partial \theta} + \rho w\frac{\partial v}{\partial z} + \frac{\rho u v}{r} = -\frac{1}{r}\frac{\partial P}{\partial \theta} \tag{3}$$

$$\rho\frac{\partial w}{\partial t} + \rho u\frac{\partial w}{\partial r} + \frac{\rho v}{r}\frac{\partial w}{\partial \theta} + \rho w\frac{\partial w}{\partial z} = -\frac{\partial P}{\partial z} \tag{4}$$

It is advantageous to formulate the energy and species conservation equations in terms of coupling functions along the lines of the Shvab-Zeldovich formulation. Consequently, linear combinations of the fuel and oxidizer species conservation equations and the energy and fuel conservation

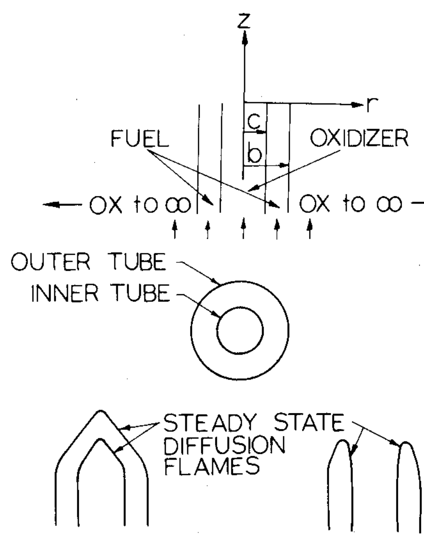

Fig. 1 Diffusion flame shapes for the two possible steady-state solutions.

equations can be written as

$$\frac{\partial}{\partial t}(\rho y)+\frac{1}{r}\frac{\partial}{\partial r}\left[r\left(\rho u y-\rho D\frac{\partial y}{\partial r}\right)\right]+\frac{1}{r}\frac{\partial}{\partial \theta}\left(\rho v y-\frac{\rho D}{r}\frac{\partial y}{\partial \theta}\right)$$

$$+\frac{\partial}{\partial z}\left(\rho w y-\rho D\frac{\partial y}{\partial z}\right)-\frac{\sigma\left[\frac{\partial P}{\partial t}+V\cdot\nabla P\right]}{\left[\sum_{j=1}^{N} h_j^\circ W_j(\nu_j'-\nu_j'')\right]}=0 \tag{5}$$

where $\sigma=0$ if $y=\beta$ and $\sigma=1$ if $y=\tau$.

Under the assumptions,

$$\frac{1}{W}=\sum_{j=1}^{N}\frac{Y_j}{W_j}=\text{const} \quad \text{and} \quad c_P=\text{const}$$

the equation of state can be expressed in terms of coupling functions as

$$PW=\frac{\rho R^\circ \tau}{c_P}\left[\sum_{j=1}^{N} h_j^\circ W_j(\nu_j'-\nu_j'')\right]$$

$$+\frac{\rho R^\circ \alpha_f}{c_P}\left[\sum_{j=1}^{N} h_j^\circ W_j(\nu_j'-\nu_j'')\right]+\rho R^\circ T^\circ \tag{6}$$

All variables will be considered to be the sum of their steady-state values and the time-dependent perturbations about these values.

Steady-State Problem

The geometry of the problem is illustrated in Fig. 1. There are two concentric tubes, of radii c and b. Both tubes terminate at $z=0$. Oxidizer flows through the inner tube and outside the outer tube, extending up to infinity. The fuel flows between the inner and outer tubes. The figure shows arrangement of the concentric tubes, the fuel and oxidizer flow, and the two types of diffusion flame shapes that can exist in the steady-state problem. The objective is to solve for all unknowns in the region $z>0$.

In addition to the assumptions of the Burke-Schumann problem, the following assumptions are made:

1) Axial energy conduction/diffusion is negligible in comparison with radial energy conduction/diffusion, i.e., $\partial\bar{T}/\partial z \ll \partial\bar{T}/\partial r$.

2) The incoming fuel and oxidizer streams are at the same temperature T_s, an assumption that can be relaxed if necessary. For convenience, the reference temperature T° is chosen to be equal to T_s.

An important assumption in this formulation is that the axial gradients of temperature and species mass fraction are small in comparison to the radial gradients. Such an assumption is questionable in the region near the flame tip. However, as the flow approaches this region, most of the heat release from the flame and most of the species consumption/creation will already have occurred. Consequently, even in this region, the assumption is not too unreasonable.

From a stability viewpoint, the assumptions $\bar{\rho}\bar{w}=\text{const}$ and $\bar{\rho}\bar{D}=\text{const}$ of the Burke-Schumann problem need careful examination. If viscosity had been included in the steady-state problem, the first of these assumptions would be impossible. In the inviscid case, the assumption remains a possible solution to the governing equations. Given the first of these assumptions, the second is not unreasonable, as one can imagine a situation in which $\bar{D}$ changes in such a way as to keep $\bar{\rho}\bar{D}$ constant. However, it must be noted that there are no strong physical grounds for either of these assumptions. On the other hand, if these assumptions are not made, the steady-state problem takes on a nonlinear character.

It must also be remarked here that, in the actual case, the flames existing just beyond the exit of the tubes are not diffusion limited, but are more premixed in character. These flames may themselves be stable or unstable. If they are stable, their primary effects are to change the initial Mach number of the flow for the diffusion flame region and to distort the velocity profiles. The present analysis will remain applicable except for a small region near the exits of the tubes. The change in initial Mach number will not qualitatively affect any of the conclusions. The effect of distortion in the velocity profile is difficult to predict. On the other hand, if these flames are unstable, they may in fact be the cause of the overall instability that is experimentally observed.

The differential equations require the specification of initial and boundary conditions for their solution. For this purpose, the mass fraction of the fuel in the fuel tube and the mass fractions of the oxidizer in the oxidizer tubes will be taken to be the constants $Y_{f,0}$ and $Y_{ox,0}$, respectively. This allows for the presence of product or neutral species in the initial unreacted gas.

It can be shown that the governing equations when applied to the steady-state problem as specified above reduce to the following system of equations:

$$\bar{\rho}\bar{w}=S=\text{const} \tag{7a}$$

$$\bar{\rho}\bar{D}=A=\text{const} \tag{7b}$$

$$\left[\frac{\bar{w}}{\bar{D}}\right]\frac{\partial\bar{y}}{\partial z}-\frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial\bar{y}}{\partial r}\right]=0, \qquad \bar{y}=\bar{\beta},\bar{\tau} \tag{7c}$$

subject to:

$\bar{y}$ bounded everywhere

$$\bar{\beta}=-\frac{Y_{ox,0}}{W_{ox}\nu_{ox}}, \quad \bar{\tau}=0 \qquad \text{at } z=0 \text{ and } 0\le r<c$$

$$\text{at } z=0 \text{ and } b\le r<\infty$$

$$\text{and for } z>0,\ r=\infty$$

$$\bar{\beta}=\frac{Y_{f,0}}{W_f\nu_f}=\bar{\tau} \qquad \text{at } z=0 \text{ and } c\le r<b$$

$$\bar{\alpha}_f=-\bar{\beta} \text{ if } \bar{\beta}>0$$

$$=0 \text{ otherwise} \tag{7d}$$

$$\bar{P}=\text{const} \tag{7e}$$

$$\bar{P}W=\frac{R^\circ\bar{\rho}\bar{\tau}}{c_P}\left[\sum_{j=1}^{N} h_j^\circ W_j(\nu_j'-\nu_j'')\right]$$

$$+\frac{R^\circ\bar{\rho}\bar{\alpha}_f}{c_P}\left[\sum_{j=1}^{N} h_j^\circ W_j(\nu_j'-\nu_j'')\right]+\bar{\rho}R^\circ T^\circ \tag{7f}$$

It should be mentioned here that Eq. (7d) is a consequence of the flame surface assumption in the Burke-Schumann problem. Its analog in the linearized unsteady case is introduced later.

The solution to the system of Eqs. (7) is presented in nondimensional form after the derivation of the simplified eigenvalue problem.

Information on how the derivatives with respect to ξ (i.e., r/b) and η (i.e., z/b) scale in comparison to each other can be obtained by considering the different terms in the steady-state differential equation. This results in the scaling law

$$\frac{\partial^2}{\partial\xi^2}\sim\frac{1}{\xi}\frac{\partial}{\partial\xi}\sim P_c\frac{\partial}{\partial\eta} \tag{8}$$

where $P_c=\bar{w}b/\bar{D}\gg 1$.

Differentiating the steady-state solution to be presented later (say, for $W\bar{\beta}$) with respect to η, it can be seen that if P_c is sufficiently large,

$$\partial/\partial\eta \sim K \tag{9}$$

where $K \ll 1$.

Linearized Unsteady Equations

All variables will be considered to be the sum of their steady-state values and the time-dependent perturbations about these values. Thus, in general,

$$x = \bar{x} + x' \tag{10}$$

It should be noted here that $\bar{u} = \bar{v} = 0$. The diffusion coefficient D is assumed to be unchanged from its steady-state value $\bar{D}$.

Substituting Eq. (10) into Eqs. (1-6) utilizing whenever appropriate the system of Eqs. (7), and neglecting all quadratic and higher-order terms in the primed quantities, the following equations are obtained.

Continuity:

$$\frac{\partial\rho'}{\partial t} + \frac{\partial\bar{\rho}u'}{\partial r} + \frac{\bar{\rho}u'}{r} + \frac{1}{r}\frac{\partial\bar{\rho}v'}{\partial\theta} + \frac{\partial}{\partial z}(\bar{\rho}w' + \rho'\bar{w}) = 0 \tag{11}$$

Momentum:

$$\bar{\rho}\frac{\partial u'}{\partial t} + \bar{\rho}\bar{w}\frac{\partial u'}{\partial z} = -\frac{\partial P'}{\partial r} \tag{12}$$

$$\bar{\rho}\frac{\partial v'}{\partial t} + \bar{\rho}\bar{w}\frac{\partial v'}{\partial z} = -\frac{1}{r}\frac{\partial P'}{\partial\theta} \tag{13}$$

$$\bar{\rho}\frac{\partial w'}{\partial t} + \bar{\rho}\bar{w}\frac{\partial w'}{\partial z} + \bar{\rho}w'\frac{\partial\bar{w}}{\partial z} + \bar{\rho}u'\frac{\partial\bar{w}}{\partial r} + \rho'\bar{w}\frac{\partial\bar{w}}{\partial z} = -\frac{\partial P'}{\partial z} \tag{14}$$

Coupling function:

$$\frac{\partial}{\partial z}\left[\bar{\rho}\bar{w}y' + \bar{y}\bar{w}\rho' + \bar{\rho}\bar{y}w' - \bar{\rho}\bar{D}\frac{\partial y'}{\partial z} - \rho'\bar{D}\frac{\partial\bar{y}}{\partial z}\right]$$

$$+ \frac{1}{r}\frac{\partial}{\partial r}\left[r\left(u'\bar{\rho}\bar{y} - \bar{\rho}\bar{D}\frac{\partial y'}{\partial r} - \rho'\bar{D}\frac{\partial\bar{y}}{\partial r}\right)\right]$$

$$+ \frac{1}{r}\frac{\partial}{\partial\theta}[\bar{\rho}\bar{y}v'] - \frac{1}{r^2}\frac{\partial}{\partial\theta}\left[\bar{\rho}\bar{D}\frac{\partial y'}{\partial\theta}\right] + \frac{\partial}{\partial t}[\bar{\rho}y' + \rho'\bar{y}]$$

$$-\frac{\sigma\left[\dfrac{\partial P'}{\partial t} + \bar{w}\dfrac{\partial P'}{\partial z}\right]}{\sum\limits_{j=1}^{N} h_j^\circ W_j(\nu_j' - \nu_j'')} = 0 \tag{15}$$

State:

$$P'W = \frac{R^\circ}{c_P}\left[\sum_{j=1}^{N} h_j^\circ W_j(\nu_j' - \nu_j'')\right]$$

$$\times(\tau'\bar{\rho} + \rho'\bar{\tau} + \rho'\bar{\alpha}_f + \alpha_f'\bar{\rho}) + \rho' R^\circ T^\circ \tag{16}$$

It can be shown that the linearized unsteady flame surface approximation result in complex notation is[6]

$$\alpha_f' = -\beta' \quad \text{if } \bar{\beta} > 0$$
$$= 0 \quad \text{otherwise} \tag{17}$$

A natural question that arises here is whether the thin-flame approximation remains valid in the unsteady case. For small perturbations, it may be expected that the thin-flame approximation is valid provided the frequency of the oscillations is not too large.

The perturbed quantities are assumed to be of the following mathematical form:

$$x' = f_x(r)e^{in\theta}e^{\alpha_z z}e^{\alpha_t t} \tag{18}$$

where $f_x(r)$ represents amplitude functions dependent on r. Since the coefficients of Eqs. (11-17) are dependent on z, it follows that the amplitude functions must also be dependent on z. However, from the steady-state solution, it is known that the coefficients are only weakly dependent on z. Hence, the amplitude functions are also weakly dependent on z. For simplicity in the notation, this is not represented. Because they are weakly dependent on z, the following approximations are made: $\partial f_x(r)/\partial r \simeq \mathrm{d}f_x(r)/\mathrm{d}r$ and $\partial f_x(r)/\partial z \simeq 0$. From this point of view, z simply becomes a parameter in the differential equations for the amplitude functions.

In Eq. (18), n is an integer, α_z an imaginary number, and α_t a complex number whose real part determines the decay or growth of the disturbances. Equation (18) can be considered to be a Fourier component of an arbitrary three-dimensional disturbance. In linear theory such components do not interact and hence can be treated separately. The amplitude functions $f_x(r)$ are subject to the restriction that they be bounded at $r=0$ and as $r \to \infty$. This restriction is equivalent to imposing boundary conditions on the differential equations.

The Simplified Eigenvalue Problem

For a given steady-state solution and for specified values of α_z and n, Eqs. (11-17) can be solved for the functional form of the amplitude functions $f_x(r)$ (in the present context, they can also be referred to as eigenfunctions) and the eigenvalue α_t. However, such a task would be both time consuming and difficult and hence it is appropriate to simplify these equations by applying the scaling laws [Eqs. (8) and (9)] to their coefficients and retaining only the more important terms. When this is done, Eqs. (11-17) with the amplitude functions as dependent variables can be written as follows.

Continuity:

$$\bar{\rho}\frac{\mathrm{d}f_u}{\mathrm{d}r} + f_u\left(\frac{\partial\bar{\rho}}{\partial r} + \frac{\bar{\rho}}{r}\right) + f_\rho(\alpha_t + \bar{w}\alpha_z) + f_w\alpha_z\bar{\rho} + f_v\frac{in\bar{\rho}}{r} = 0 \tag{19}$$

Momentum:

$$\frac{\mathrm{d}f_P}{\mathrm{d}r} + f_u(\bar{\rho}\alpha_t + \bar{\rho}\bar{w}\alpha_z) = 0 \tag{20}$$

$$\frac{inf_P}{r} + f_v(\bar{\rho}\alpha_t + \bar{\rho}\bar{w}\alpha_z) = 0 \tag{21}$$

$$\alpha_z f_P + f_w(\bar{\rho}\alpha_t + \bar{\rho}\bar{w}\alpha_z) + f_u\bar{\rho}\frac{\partial\bar{w}}{\partial r} = 0 \tag{22}$$

Coupling function:

$$f_y\left(\bar{\rho}\bar{w}\alpha_z + \frac{\bar{\rho}\bar{D}n^2}{r^2} + \bar{\rho}\alpha_t\right) + \bar{\rho}\bar{y}\frac{\mathrm{d}f_u}{\mathrm{d}r} + f_u\frac{1}{r}\frac{\partial r\bar{\rho}\bar{y}}{\partial r} + f_v\frac{\bar{\rho}in\bar{y}}{r}$$

$$+ f_\rho(\bar{y}\bar{w}\alpha_z + \alpha_t\bar{y}) + f_w\bar{\rho}\bar{y}\alpha_z - \frac{\sigma f_P(\alpha_t + \bar{w}\alpha_z)}{\sum\limits_{j=1}^{N} h_j^\circ W_j(\nu_j' - \nu_j'')} = 0 \tag{23}$$

State:

$$f_P W=\frac{R^\circ}{c_P}\left[\sum_{j=1}^{N} h_j^\circ W_j(\nu_j'-\nu_j'')\right](f_\tau\bar{\rho}+f_\rho\bar{\tau}+f_\rho\bar{\alpha}_f)$$
$$+\frac{\bar{\rho}R^\circ}{c_P}\left[\sum_{j=1}^{N} h_j^\circ W_j(\nu_j'-\nu_j'')\right]\begin{pmatrix}-f_\beta \text{ if } \bar{\beta}>0\\ 0 \text{ otherwise}\end{pmatrix}+f_\rho R^\circ T^\circ \tag{24}$$

It can be seen that the linearized unsteady flame surface approximation result has now been incorporated into the equation of state.

The above simplifications require some discussion. It can be seen that the order of the system of simplified equations (19-24) is lower than the order of the system of exact equations (11-17). Thus, there exists the possibility that some important properties of the stability problem may have been lost in the simplification process. This question will be discussed again.

The boundary conditions on the system of simplified equations are simply that the amplitude functions $f_x(r)$ remain bounded as $r\to\infty$ and at $r=0$.

The system of simplified equations can be combined to give a single equation in a single dependent variable. In addition, the boundedness requirements on all the amplitude functions can be transferred into conditions in terms of this single dependent variable. Choosing f_P as the single dependent variable, it can be shown that the differential equation for f_P, its boundary conditions, and the various steady-state terms occurring in the coefficients of the differential equation can be expressed in nondimensional form as

$$\frac{d^2 f_P}{d\xi^2}+\left\{\frac{(G+VF)}{[G+VF+(n^2V/P_c\xi^2)]}\frac{1}{V}\frac{\partial V}{\partial\xi}-\frac{2F}{(G+VF)}\frac{\partial V}{\partial\xi}+\frac{1}{\xi}\right\}\frac{df_P}{d\xi}+\left\{F^2-\frac{n^2}{\xi^2}-\frac{M}{V}(G+VF)^2+\frac{[(\gamma-1)/\gamma]M(G+VF)^3}{V[G+VF+(Vn^2/P_c\xi^2)]}\right\}f_P=0 \tag{25}$$

subject to:

1) As $\xi\to\infty$, f_P bounded for all n.

2) As $\xi\to 0$, $f_P\to 0$ at least as fast as ξ^2 for $n\neq 0$.

3) As $\xi\to 0$, f_P must be bounded, $df_P/d\xi\to 0$ at least as fast as ξ for $n=0$.

where

$$W\bar{\beta}=-B+(E+B)\int_0^\infty [J_1(q)-aJ_1(aq)]J_0(q\xi)e^{-q^2\eta/P_c}dq$$

$$W\bar{\tau}=(W\bar{\beta})_{B=0}$$

$$V=R\left[W\bar{\tau}+W\bar{\beta}\begin{pmatrix}-1 \text{ if } W\bar{\beta}>0\\ 0 \text{ otherwise}\end{pmatrix}+\frac{1}{R}\right]$$

$$B=\frac{WY_{ox,0}}{W_{ox}\nu_{ox}},\quad E=\frac{WY_{f,0}}{W_f\nu_f},\quad F=\alpha_z b,\quad G=\frac{\alpha_t b}{V_0},$$

$$V=\frac{\bar{w}}{V_0},\quad M=\frac{WV_0^2}{R^\circ T_s}=\frac{SV_0}{\bar{P}},\quad R=\sum_{j=1}^{N}\frac{h_j^\circ}{c_P T^\circ}\frac{W_j}{W}(\nu_j'-\nu_j'')$$

Subject to its boundary conditions, Eq. (25) is the simplified eigenvalue problem. All quantities in Eq. (25) are considered known except the eigenfunction f_P and the eigenvalue G.

The Case of $n=0$

Further discussion in this paper is restricted to the case of $n=0$. The author has not been successful in solving the eigenvalue problem for $n\neq 0$. The reason for this is not clear. However, it will be seen later that almost every disturbance with $n=0$ is unstable. Consequently, the disturbances under investigation here are likely to represent some of the most unstable.

When $n=0$, Eq. (18) implies that the perturbed quantities are not functions of θ. Therefore, this case can be considered to represent only axial disturbances.

An examination of Eq. (25) and its boundary conditions when $n=0$ shows that:

1) If (G_r, G_i) and (f_{Pr}, f_{Pi}) is a solution of Eq. (25) subject to its boundary conditions, then $(-G_r, G_i)$ and $(f_{Pr}, -f_{Pi})$ is also a solution. (This means that the existence of a self-excited oscillation implies the existence of a damped oscillation and vice versa.)

2) If for a given F_i, (G_r, G_i) and (f_{Pr}, f_{Pi}) is a solution of Eq. (25) subject to its boundary conditions, then for the case of $-F_i$, the solution is $(G_r, -G_i)$ and $(f_{Pr}, -f_{Pi})$.

The phase velocity of the disturbances can be obtained by considering the equation of the constant-phase path in Eq. (18). For the case $n=0$, this is $\alpha_{zi}z+\alpha_{ti}t=\text{const}$. Differentiating with respect to t and nondimensionalizing α_{zi} and α_{ti}, the phase velocity in the z direction is obtained as

$$\frac{1}{V_0}\frac{dz}{dt}=-\frac{G_i}{F_i}$$

When considered with the result previously obtained, this leads to the conclusion that the waves propagate in only one direction (either $+z$ or $-z$). It does not seem possible to obtain the direction of propagation only from analytical considerations. However, the numerical solution of the eigenvalue problem gives negative values of G_i for positive F_i. Hence, waves propagate only in the positive z direction (i.e., the downstream direction).

3) It can be shown that as $\xi\to 0$, the asymptotic solution of Eq. (25) for the case $n=0$ satisfying the boundary condition is

$$f_P=A_1 \tag{26a}$$

$$\frac{df_P}{d\xi}=-\frac{A_1 C^{\circ 2}\xi}{2} \tag{26b}$$

where $C^{\circ 2}=F^2-(M/V^\circ\gamma)(G+V^\circ F)^2$.

4) For $\xi\geq 5.5$, it is found that the steady-state solutions become practically independent of ξ. In particular, $V\simeq 1$ and $\partial V/\partial\xi\simeq 0$ for a wide range of values of the parameters that determine the steady-state solution. Under these conditions, it can be shown that the asymptotic solution of Eq. (25) for the case $n=0$, satisfying the boundary condition, is

$$f_P=A_2H_0^{(1)}(Z) \qquad \text{if } 0<\arg Z<\pi \tag{27a}$$

$$f_P=A_3H_0^{(2)}(Z) \qquad \text{if } -\pi<\arg Z<0 \tag{27b}$$

$$f_P=A_4J_0(Z)+A_5Y_0(Z) \text{ if } Z \text{ is real} \tag{27c}$$

where

$$Z=[F^2-(M/\gamma)(G+F)^2]^{1/2}\xi$$

It should be mentioned here that the above expression for Z gives, for a given value of ξ, two possible values of Z, one lying above the real axis and the other below. From Eq. (27) it would appear that the aysmptotic solution will be different, depending on which of the two values of Z is chosen. However, it can be shown that the two asymptotic solutions

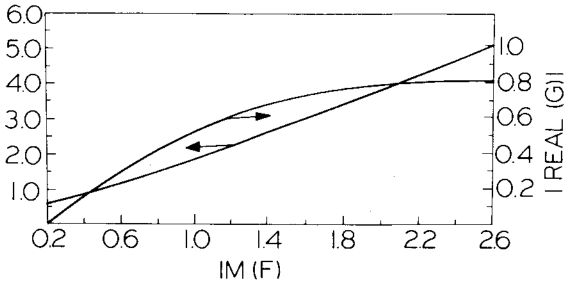

Fig. 2 Dependence of G on inverse measure of wavelength of axisymmetric disturbance.

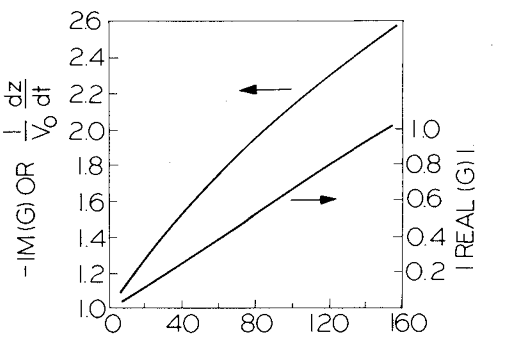

Fig. 3 Dependence of G and nondimensional phase velocity on heating value of the chemical reaction for axisymmetric disturbances.

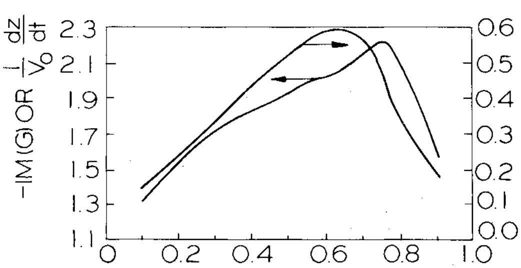

Fig. 4 Dependence of G and nondimensional phase velocity on the "thinness" of the fuel zone for axisymmetric disturbances.

differ only by a multiplicative constant and hence are completely equivalent.

It has not been possible to solve the eigenvalue problem analytically. Numerical solutions have been obtained by integrating Eq. (25) from both ends of the region $0<\xi<5.5$ using Eqs. (26) and (27) for initialization and matching the numerical solutions at $\xi=2.75$. The integrations were performed using the Runge-Kutta method of order two. As the differential equation is of the second order, the matching condition at $\xi=2.75$ is that the function and its derivative are continuous at this point. Noting that Eqs. (26) and (27) each contain an arbitrary constant, both of which were assigned the value unity for purposes of starting the integration, the matching condition is that the function and its derivative obtained from one side are each a multiplicative constant times the function and derivative obtained from the other side. These two relations can be used to eliminate the multiplicative constant, resulting in the condition that the quotient of the function and its derivative from one side is equal to the quotient of the function and its derivative from the other side. This relation can be considered as a nonlinear algebraic equation in G and can be numerically solved by iteration. Various step sizes for the integration and various matching points were used to insure consistency of the results.

The dependence of G and $(1/V_0)(\mathrm{d}z/\mathrm{d}t)$ on the parameters F, R, and a are presented in Figs. 2-4. In these plots,

Table 1 Typical values of parameters and their range of variation

Parameter	Typical value	Range over which parameter was varied
a	0.5	0.1-0.9
F	(0.0,1.0)	(0.0,0.2)-(0.0,2.6)
n	0	–
P_c	100[a]	–
R	77.5[b]	7.0-154.0

[a]Estimated from $(\hat{w}b)/\hat{D}$ with $\hat{w}=5$ m/s, $b=2\times10^{-4}$ m, $\hat{D}=1\times10^{-5}$ m^2/s (values believed representative near burning surface of AP composite solid propellants). [b]Estimated for the reaction $CH_4+2O_2\rightarrow CO_2+2H_2O$ at a standard temperature $T^\circ=298$ K.

all parameters have typical values except when they are being varied. Typical values of the parameters along with the ranges over which they are varied are shown in Table 1.

The main conclusions from these plots can be summarized as outlined in the following paragraphs.

As the wavelength of the disturbance decreases, the energy interaction between the flame and the disturbance increases, leading to larger growth rates. The frequency of the disturbance also increases with decreasing wavelength.

The growth rate of the disturbance increases almost linearly with the magnitude of the heat release from the flame and it appears that heat release from the flame is a necessary condition for instability. The frequency of oscillation and the phase velocity also increase with increasing heat release from the flame.

The curves of the growth rate and the frequency of oscillation of the disturbance attain maxima at different values of a. The shape of these curves in Fig. 4 probably results from a complex interaction between many factors. Some of the factors that almost certainly play a part are: the geometry of the problem, i.e., whether circular or planar; the distance (b-c) between the diffusion flames arising, respectively, at the edge of the first and second tubes; and the total exothermicity of the diffusion flames as determined by the area of the annulus between the tubes. It is important to note that Fig. 4 predicts instability for all values of a $(0\leq a\leq 1)$.

In order to make the results of Fig. 4 applicable for planar geometries, it becomes necessary to proceed to certain geometrical limits. While doing so, one must be careful to ensure that the Peclet number formed with the minimum characteristic dimension is always large compared to one. Bearing this in mind, some special cases are discussed below.

If (b-c) is chosen such that the Peclet number formed with this dimension is large compared to unity and if now b and c are allowed to increase indefinitely in magnitude so that $a\rightarrow 1$, the case would correspond to a planar oxidizer/fuel/oxidizer sandwich. In this case, Fig. 4 predicts instability (although with reduced growth rates compared to circular geometries) and, in addition, also predicts that the growth rate is independent of (b-c), i.e., the growth rate is independent of the fuel slot thickness. Possibly in this case, the effects of interflame distance and total exothermicity cancel each other, resulting in a growth rate that is independent of the fuel slot thickness.

A different geometrical limit can be obtained by choosing c so that the Peclet number formed with this length scale is large compared to unity and then increasing b indefinitely so that $a\rightarrow 0$. The interflame distance becomes indefinitely large. Figure 4 once again predicts instability with a reduced growth rate that is independent of the exact values of b and c.

However, no general conclusions can be drawn regarding instability as a function of problem geometry, as the following example will show. If $c=0$, then $a=0$ and the only characteristic dimension in the problem is b. In this case, there is only a single diffusion flame arising at the interface between the fuel and oxidizer flows. As the value of b is increased, the geometry changes from circular to planar. Figure 4 predicts a growth rate that is independent of the value of b, i.e., the growth rate is independent of whether the geometry is circular or planar. Perhaps in this case, the lowered growth rate found in Fig. 4 can be attributed only to the fact that there is a single diffusion flame. For large b, this case can be considered to represent the diffusion flame arising at the interface between two semi-infinite slabs of fuel and oxidizer.

Effects of Unsteady Mass and Energy Diffusion and Unsteady Energy Conduction

The simplified eigenvalue problem [Eq. (25) and its boundary conditions] was obtained by assuming that the value of P_c was large compared to unity. This is equivalent to neglecting the effects of unsteady mass and energy diffusion and energy conduction. (The Lewis number is unity in this formulation.) If this assumption had not been made, the differential equation representing the eigenvalue problem would have been of fourth order. Thus, there arises the question as to whether or not solutions of the simplified eigenvalue problem are true asymptotic representations of the solutions of the unsimplified differential equations in the limit of large P_c.

It can be shown that the nondiffusion and nonconduction asymptotic solutions obtained here [Eqs. (26) and (27)] can also be recovered from the exact equations in the limit $P_c \rightarrow \infty$. This encourages the author to believe that an exact analysis of the eigenvalue problem will still find axisymmetric diffusion flames unstable to axisymmetric disturbances. However, the diffusion-conduction solutions of the exact equations cannot be recovered and these solutions will alter the conclusion of the simplified eigenvalue problem that the existence of self-excited oscillations implies the existence of damped oscillations and vice versa.

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